DIVERGENT SERIES AND L-FUNCTIONS

ASVIN GOTHANDARAMAN

ABSTRACT. The goal of this note is to develop a theory of summing divergent series that is applicable to the Riemann zeta function and Dirichlet L-functions. In particular, I will prove the rationality of the values of Dirichlet L-functions at negative integers, their compatibility with the Galois action, generalized Kummer congruences for Dirichlet L-functions and other p-adic information on these values and finally compute the values explicitly in terms of generalized Bernoulli numbers.

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The results in this paper are mostly based off of lectures notes of Prof. Akshay Venkatesh[1].

The only completely original section is the one on the computation of special values in terms of generalized Bernoulli numbers. Parts of the section on Kummer congruences are also original as well as the subsection on compatibility with Galois action.

1. INTRODUCTION

Recall the standard definitions:

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

[1]Section 3, The Analytic Class Number Formula and L-functions

and for a Dirichlet character χ of conductor f:

$$L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}$$

both of which are defined for $\mathrm{Re}(s) > 1$ and can be meromorphically continued to the entire real plane (except possible a pole at $s = 1$.

In particular, the value of the series at negative integers is computed by the often opaque process of analytic continuation and results at these points are established by analytic continuation.

However, it turns out that L-function have striking arithmetic properties at negative integers compared to the positive integers. For instance, $\zeta(-k) \in \mathbb{Q}$ for all $k \geq 0$ while the values at positive integers have a transcendental factor.

We would like to develop tools to work with the negative integers directly. For instance, we would like to justify the following "proof" of the Kummer congruences rigorously.

Theorem 1 (Kummer Congruence:). *Suppose that k, l are two positive integers such that $k \equiv l \pmod{p-1}$ and neither one is congruent to 1. Then, $\zeta(-k), \zeta(-l)$ are in $\mathbb{Z}_p$ and $\zeta(-k) \equiv \zeta(-l) \pmod{p}$.*

Proof. Let us suppose that we can treat $\zeta(-k) = 1^k + 2^k \dots$ as an actual convergent series. Then we might attempt the following proof:

Under the conditions on k, l, $n^k \equiv n^l \pmod{p}$ for all p and so their summations are also congruent! □

Clearly this proof is completely bogus in the standard theory of series for many reasons. Nevertheless, the theory we develop will allow us to make this argument rigorous with very little modifications! Even more, it will generalize naturally to all the Dirichlet L-series.

Similarly, consider the following evaluation of $\zeta(-k)$:

Write out each "series expansion" as

$$\begin{aligned}
\zeta(0)\frac{t^0}{0!} \text{ "} = \text{" } & 1^0\frac{t^0}{0!} + 2^0\frac{t^0}{0!} + 3^0\frac{t^0}{0!} + \dots \\
\zeta(1)\frac{t^1}{1!} \text{ "} = \text{" } & 1^1\frac{t^1}{1!}! + 2^1\frac{t^1}{1!} + 3^1\frac{t^1}{1!} + \dots \\
\zeta(2)\frac{t^2}{2!} \text{ "} = \text{" } & 1^2\frac{t^2}{2!} + 2^2\frac{t^2}{2!} + 3^2\frac{t^2}{2!} + \dots \\
& \vdots \\
\zeta(3)\frac{t^3}{3!} \text{ "} = \text{" } & 1^k\frac{t^k}{k!} + 2^k\frac{t^k}{k!} + 3^k\frac{t^k}{k!} + \dots \\
& \vdots
\end{aligned}$$

and instead of summing horizontally, let us sum vertically first. The left hand side is the exponential generating series for $\zeta(k)$, let us call it $f(t)$. The right hand side sums to e^{nt} for

$n = 1, 2 \ldots$. That is, we get:

$$\sum_{k\geq 0} \zeta(k) \frac{t^k}{k!} \text{''} = \text{''} e^t + e^{2t} + e^{3t} + \ldots \text{''} = \text{''} \frac{e^t}{1-e^t}.$$

The right hand side is the exponential generating series for the Bernoulli numbers (with some minor changes) and on equating the coefficients of t^k, one obtains $\zeta(-k) = -B_{k+1}/(k+1)$ which is the right answer! Once again, our theory will make this rigorous and even extend the method to all Dirichlet L-functions.

Let me now describe a brief, imprecise outline of the note:

First, we will define an "abstract sequence space" over a field K which is simply the set of sequences that eventually satisfy a linear recurrence. In particular, this contains all finite sequences.

We would like to define a linear functional on this space that extends the usual summation on finite sequences while maintaining some properties of the usual summations. In particular, we will want our summation to be invariant under shifting all the terms to the right and a "dilation" operator, to be defined later. In fact, we will show that under mild conditions, there is exactly one such functional.

Now, suppose that our ground field is $\mathbb{C}$. For a sequence $n \longrightarrow f(n)$ in our abstract sequence space, the zeta function is defined by:

$$\zeta_f(n) = \sum_{n\geq 1} \frac{f(n)}{n^s}$$

for $\mathrm{Re}(s) \gg 0$. I will show that there is a meromorphic continuation to the entire plane and that $s = 0$ is not a pole. Thus, we can define an operator by $f \longrightarrow \zeta_f(0)$.

We will show that this operator satisfies the invariance properties we required of our abstract summation operator. Therefore, it has to match our previously defined operator. This done, we will forget entirely about analytic continuation and deal purely with our abstractly defined operator and make the above arguments rigorous rather easily.

2. Abstract Sequence Spaces

We will define a vector space of sequences over which it will be possible to define a unique summation operator that extends the usual summation for finite sequences. The definition is as follows:

Fix a field k throughout unless otherwise specified. In this article, $\mathbb{N}$ will always refer to the set $\{1, 2, 3 \ldots\}$.

Definition 1. *Let Y^c be the set of sequences with compact support. That is, all but finitely many terms are zero.*

Then, define X to be the set of sequences that satisfy a linear recurrence eventually with a finite number of initial values. That is, $f \in X$ if and only if there exist $a_0, a_1, \ldots, a_r \in k$ such that

$$a_0 f(n) + a_1 f(n-1) + \cdots + a_r f(n-r) = 0 \text{ for all } n \gg 0.$$

All the sequences we are interested in will be of this form.

We will eventually define an extension of Σ to all of X. However, first let us provide an alternate characterization for sequences that satisfy a linear recurrence. The following is well known:

Theorem 2. *A sequence $f : \mathbb{N} \longrightarrow \overline{k}$ satisfies a linear recurrence precisely when it is a linear combination of sequences of the form $\alpha^n p(n)$ for $\alpha \in \overline{k}$ and $p(n) \in k[x]$ with the first r values being arbitrary. Furthermore, if f satisfies the linear recurrence:*

$$a_0 f(n) + a_1 f(n-1) + \cdots + a_r f(n-r),$$

then the α that appear in the linear combination are all exactly the roots of

$$a(t) = a_0 t^r + a_1 t^{r-1} + \cdots + a_r = 0$$

and the degree of the polynomial $p(n)$ associated to such an α is equal to the multiplicity of α in $a(t)$ minus 1.

Definition 2. *Let us then define Y to be the space of sequence of the above form taking values in k. That is. Y consists of sequences of the form:*

$$f(n) = \sum_{\alpha,k} c_k \alpha^n n^k \in k \qquad \text{for } \alpha \in \overline{k}^{\times}, c_k \in k^{\times} \text{ and } k \in \mathbb{N}.$$

For $f \in X$, we will have such a decomposition (in a unique way) for large n. The α that appear will be called the exponents of f and the variable α will be reserved for this purpose.

An important consequence of the theorem is:

Lemma 3. *The sum and product of sequences in X is again part of X. That is, for $f, g \in X$ the sequences $n \longrightarrow f(n) + g(n)$ and $n \longrightarrow f(n)g(n)$ are both again in X.*

It will be convenient to break X up into into the following subspaces.

Definition 3. *Let us define Y^α to be the vector space of sequences of the form $\alpha^n p(n)$ for $p(n) \in k[x]$ and $Y^{\neq 1}$ to be the complement of Y^1. Similarly, we can define $X^\alpha = Y^\alpha + Y^c$.*

2.1. **Operators on Sequences.** We will also make heavy use of the following two endomorphisms of X:

Define $S : X \longrightarrow X$ by $S[f](n) = f(n-1)$ for $n \geq 2$ and $S[f](1) = 0$. This is the "shift by one to the right" operator. $S[f]$ is in X since it satisfies the same eventual recurrence as f.

Next, for $m \in \mathbb{N}$, define $D_m : X \longrightarrow X$ by $D_m[f](n) = f(n/m)$ if $m|n$ and 0 otherwise. This is the dilation operator and it simply stretches our sequence out. It is not hard to see that $D_m[f]$ also eventually satisfies a linear recurrence.

What will be crucially important for us is that for any $f \in Y$, the span of f under S is finite dimensional. This is in fact equivalent to saying that f satisfies a linear recurrence.

Equivalently, for any $f \in X$, we can find a polynomial $P_f \in k[x]$ such that $P_f(S)[f] \in Y^c$.

3. Extending the Summation Operator

Finally, we are in a position to realize our goal of defining a summation for all sequences in X. The main theorem of this section is the following:

Theorem 4. *There is a unique linear functional $\Sigma : X \longrightarrow k$ such that:*

(1) The restriction $\Sigma : Y^c \longrightarrow k$ is the usual summation operator on finite sequences.
(2) For $f \in Y^{\neq 1}$, the operator is invariant under S. That is, $\Sigma(S[f]) = \Sigma(f)$.
(3) For all $f \in X$ and $m \in \mathbb{N}$, the operator is invariant under D_m. That is, $\Sigma(D_m[f]) = \Sigma(f)$.

In fact, as we will see the proof of this theorem will be readily adapted to prove stronger versions of this theorem as we need. See the remark following the proof for stronger versions of this theorem.

The idea of the proof is simple. The invariance under the various operators will force uniqueness. For instance, suppose $f \in Y^\alpha$ and $\alpha \neq 1$. Recall that the orbit of f under S is finite modulo Y^c. In other words, for the characteristic polynomial $P_f(X) \in k[X]$ of f, we have:

$$P_f(S)[f] = f^c \qquad \text{for some } f^c \in Y^c.$$

Applying Σ to both sides and using the invariance under Σ, we are forced into:

$$P_f(1)(\Sigma(f)) = \Sigma(f^c)$$

and since the eigenvalues α of f are not 1, $P_f(1) \neq 0$ and this determines the value of $\Sigma(f)$. A similar argument holds for $f \in Y^1$ using D_m instead.

Of course, we still have to establish that this defines a linear functional. Instead, we will take a different approach that proves uniqueness and existence simultaneously. We will first extend the operator to $X^{\neq 1}$ from Y^c and then use this to further extend to all of X.

Proof. (2) Note that an S-invariant functional on a vector space W is the same as a functional on $W/(S-1)W$. However, I claim that the inclusion $Y^c \longrightarrow X^\alpha$ for $\alpha \neq 1$ induces an isomorphism $Y^c/(S-1)Y^c \cong X^\alpha/(S-1)X^\alpha$. This is clearly sufficient to prove (2).

Consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y^c & \longrightarrow & X^\alpha & \longrightarrow & X^\alpha/Y^c & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (S-1)} & & \downarrow{\scriptstyle (S-1)} & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & Y^c & \longrightarrow & X^\alpha & \longrightarrow & X^\alpha/Y^c & \longrightarrow & 0 \\
 & & \twoheaddownarrow & & \twoheaddownarrow & & & & \\
 & & Y^c/(S-1)Y^c & & X^\alpha/(S-1)X^\alpha & & & &
\end{array}$$

The final vertical arrow is an isomorphism because the eigenvalues of $(S-1)$ are all non zero. Applying the snake lemma induces the required isomorphism $Y^c/(S-1)Y^c \cong X^\alpha/(S-1)X^\alpha$.

(3) We will follow the same strategy, only making the replacements $(Y^c, X^\alpha, S-1) \longrightarrow (X^{\neq 1}, X, D_l - 1)$ for some prime $l \neq 0 \in k$.

First, let us prove that $\Sigma(D_m[f]) = \Sigma(f)$ for $f \in X^\alpha, \alpha \neq 1$. This follows because $S^m D_m = D_m S$. We know that there exists $f^c \in Y^c$ such that $(S-\alpha)^n[f] = f^c$ and therefore $\Sigma(f) = \Sigma(f^c)/(1-\alpha)^n$.

On the other hand:

$$D_m[f^c] = D_m(S-1)^n[f] = (S^m - 1)^n[D_m[f]]$$

and since Σ is D_m invariant on Y^c, $\Sigma(D_m[f]) = \Sigma(f^c)/(1-\alpha)^n$ and Σ is therefore D_m invariant on $X^{\neq 1}$.

Note that $X/X^{\neq 1}$ can be identified with the space of all polynomials. This vector space clearly has the basis $n, n^2, \ldots$ where n^k stands for the sequence $f(n) = n^k$. To mimic the previous proof, we need to prove that D_m does not have eigenvalue 1 on this vector space.

However, this follows from the following observation:

$$D_m[n^k] = \frac{1 + \zeta_l^n + \zeta_l^{2n} + \cdots + \zeta_l^{(l-1)n}}{l} \left(\frac{n}{l}\right)^k \equiv \frac{n^k}{l^{k+1}} \pmod{X^{\neq 1}}.$$

Therefore, the n^k form a basis of eigenvectors with eigenvalues $l^{-(k+1)} \neq 1$. After this step, the rest of the proof proceeds exactly as before after making the replacements.

$\square$

Remark 5. *A couple of remarks about strengthening the previous theorem:*

(1) Note that, even if we only insist on S-invariance, there is only a unique extension of $\Sigma : X^\alpha \longrightarrow k$ for each $\alpha \neq 1$.
(2) Second, to extend the operator to X^1, we only need to first extend Σ to $Y^{|\alpha|=1}$ as can be seen by examining the proof.
(3) As noted before the proof, for $f \in X^{\neq 1}$, the value of $\Sigma[f]$ is $\Sigma(f^c)/P_f(1)$ where $f^c \in Y^c$ and P_f is the characteristic polynomial of S on the span of f in X/Y^c.

All three of these remarks will be used in what follows.

4. Some Examples

Let us see how we can use these ideas to compute $\Sigma(f)$ for a few sequences.

Example 1. *Let $f(n) = \alpha^n$ for $\alpha \neq 1$. This is the analogue of the usual geometric series $\alpha + \alpha^2 + \ldots$ and we should expect the same answer. Indeed:*

Note that $S[f](n) = \alpha^{n-1}$ for $n \geq 2$. Therefore, $S[f](n) = \alpha^{-1} f(n)$ for $n \geq 2$. Let $g = (\alpha, 0, 0, \ldots) = f - \alpha S[f]$ Since Σ is invariant under S in this case, we have:

$$\alpha = \Sigma(g) = (1 - \alpha)\Sigma(f)$$

and so $\Sigma(f) = \alpha/(1 - \alpha)$

Example 2. *Now let $f(n) = 1$ be constant. In this case, we should expect the value to be equal to $\zeta(0)$. However, in this case, we cannot use the translation operator since $\alpha = 1$. Instead, let us use D_2.*

Let $g = 2D_2[f]$. This sequence goes as $0, 2, 0, 2, 0, 2 \ldots$. Therefore, $f(n) - g(n) = -(-1)^n$ and we are in the case of our previous example. Applying Σ and using invariance, we have:

$$\Sigma(f) - 2\Sigma(f) = 1/2 \implies \Sigma(f) = -1/2$$

and we see that $\Sigma(f)$ is indeed equal to $\zeta(0)$.

Example 3. *Now let $f(n) = n$. In this case, we should expect the value to be equal to $\zeta(1)$. We will need to use D_2 once again.*

Let $g = D_2[f] = \frac{1+(-1)^n}{2} \frac{n}{2}$. This is because $\frac{1+(-1)^n}{2}$ is 0 when n is odd and 1 when n is even. We used the same trick in the proof of Theorem 4.

Therefore, $h(n) = f(n) - 4g(n) = -(-1)^n f(n) = -(-1)^n n$ and $\alpha \neq 1$. We can now use the translation operator.

We have $S[h](n) = (-1)^n(n - 1)$ for $n \geq 2$ and therefore:

$$h(n) + S[h](n) = \begin{cases} 1 & n = 1 \\ -(-1)^n & n \geq 2 \end{cases} = -(-1)^n.$$

Using our previous examples, we quickly compute:

$$1/2 = 2\Sigma(h) = -6\Sigma(f) \implies \Sigma(f) = -1/12.$$

Once again we get the expected answer. However, further computations quickly get quite cumbersome and we will need a different method to compute $\zeta(n)$ in general.

Let us also record the exponents of a (non trivial) Dirichlet character for future use.

Example 4. *Let $\chi : \mathbb{Z}/f\mathbb{Z} \longrightarrow \mathbb{C}^\times$ be a non trivial Dirichlet Character and let $g(n) = \chi(n)$ define $g \in X$. The exponents of f are among the f-th roots of unity and distinct from 1.*

This follows from the fact that $n \longrightarrow \zeta_f^m$ (for ζ_f a primitive f root of unity) form a spanning set for the set of functions $(\mathbb{Z}/f\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ by finite Fourier analysis. Furthermore, if 1 were an exponent, this would contradict the fact that the sum of $\chi(n)$ over a period is 0.

5. Analytic continuation as divergent summation

We can finally connect back to number theory in this section. I will show that the usual process of analytic continuation and evaluation at an integer can be seen as a linear operator that satisfies the criteria of Theorem 4 (that characterizes the Σ operator).

Thus, we can work with the abstract Σ operator instead of dealing with the intricacies of complex analysis. Even more usefully, we can change our field from $\mathbb{C}$ to something more arithmetically useful. We will make crucial use of this technique soon.

Let us now take the base field to be $\mathbb{C}$. Let $f \in X$ such that it's exponents α all satisfy $|\alpha| \leq 1$. Call the set of such f to be $X^{\leq 1}$. Define the associated zeta function by:

$$\zeta_f(s) = \sum_{n \geq 1} \frac{f(n)}{n^s}.$$

Since $|\alpha| \leq 1$, $f(n)$ is bounded by a polynomial in n and therefore $\zeta_f(s)$ converges for s large enough. The first order of business will be to show that $\zeta_f(s)$ has a meromorphic continuation to the plane.

Lemma 6. *Let f and $\zeta_f(s)$ be as above. Then, $\zeta_f(s)$ has a meromorphic continuation to the plane. Moreover, it does not have a pole at $s = 0$.*

Proof. Any such f can be written as the sum of a polynomial (in n) and a $f' \in X^{\neq 1}$. Since the case of f a polynomial reduces to the usual zeta function, we only need to deal with $f' \in X^{\neq 1}$.

Since the eigenvalues of $f' \in X/X^c$ under S are not equal to 1 and the span of f under S is finite, $S - 1$ is invertible on X/X^c. That is, there exists $f'' \in X$ such that:

$$f' = (S-1)f'' + h \text{ for } h \in Y^c.$$

It is easy to see by computation that the exponents α of f'' are a subset of those of f. Since for f, $|\alpha| \leq 1$, f' is bounded by a polynomial and moreover, f'' is bounded by the *same* polynomial upto a constant factor that f' is bounded by. Converting the above equation into Dirichlet series:

$$\sum_{n \geq 1} \frac{f'(n)}{n^s} = \sum_{n \geq 1} f''(n) \left(\frac{1}{(n+1)^s} - \frac{1}{(n)^s} \right) + g(s)$$

where g is a finite sum of exponentials (and hence defined everywhere).

Now, the point is that the series on the right converges faster than the one on the left. In fact, we have the estimate:

$$\left(\frac{1}{(n+1)^s} - \frac{1}{(n)^s}\right) \sim \frac{1}{n^{s+1}}.$$

Since both f' and f'' are bounded by the same polynomial, we have shifted the region of convergence by at least 1 to the left. We can repeat this process to move the region of convergence to include 0.

Finally, since f is bounded by a polynomial, we can bound $\zeta_f(s)$ by a finite sum of zeta series of the form $\zeta(s-k)$ for varying k. Since all these series converge at $s = 0$, so does $\zeta_f(s)$.

□

Remark 7. Note that the proof above also shows that $\zeta_f(s)$ has a pole if and only if some exponent of f is equal to 1.

The lemma above gives us one way of making sense of $\zeta_f(0)$ for $f \in X^{\leq 1}$. Our next goal is to prove that the operator $f \longrightarrow \zeta_f(0)$ behaves as the operator Σ does. In particular, it satisfies the hypothesis of Theorem 4 (that characterizes the Σ operator).

Theorem 8. *The operator $X^{\leq 1} \longrightarrow \mathbb{C} : f \longrightarrow \zeta_f(0)$ satisfies the following three properties:*

(1) The restriction $\zeta_f(0) : Y^c \longrightarrow \mathbb{C}$ is the usual summation operator on finite sequences.
(2) For $f \in Y^{\neq 1} \cap X^{\leq 1}$, the operator is invariant under S.
(3) For all $f \in X \leq 1$ and $m \in \mathbb{N}$, the operator is invariant under D_m. That is, $\Sigma(D_m[f]) = \Sigma(f)$.

Proof. One can check easily that $X^{\leq 1}$ is invariant under both the operators S and D_m.

Property one is clear since for f compact, $\zeta_f(s)$ is a finite sum of exponentials. Property 3 is almost as easy. Let $g = D_m[f]$. Then:

$$\zeta_g(s) = \sum_{n \geq 1} \frac{f(nm)}{(nm)^s} = m^{-s}\zeta_f(s)$$

and at $s = 0$, we have $\zeta_g(0) = \zeta_f(0)$.

Property two is a little harder to verify. For $f \in Y^{\neq 1} \cap X^{\leq 1}$ $f = S[f]$ and now consider $\zeta_g(s) - \zeta_f(s)$. This has the series expansion:

$$\zeta_g(s) - \zeta_f(s) = \sum_{n \geq 1} \frac{f(n)}{n^s}\left(\frac{1}{(1+1/n)^s} - 1\right).$$

We can expand the term in the brackets using it's Taylor series as:

$$\left(\frac{1}{(1+1/n)^s} - 1\right) = -\frac{s}{n} + \frac{-s(-s-1)}{2!n^2} + \cdots + R_k(s,n).$$

The point is that each term is divisible by s and so vanishes at $s = 0$. To make it rigorous, let k be such that $|f(n)| \leq Cn^{k-2}$ for some constant C. This implies that $\zeta_f(s)$ is absolutely convergent for $\mathrm{Re}(s) \geq k$. This is what goes wrong for $\alpha = 1$. Consider the Taylor expansion:

$$\frac{1}{s}\left(\frac{1}{(1+1/n)^s} - 1\right) = -\frac{1}{n} + \cdots + \frac{P_{k-1}}{n^{k-1}} + R_k(s,n)$$

where P_{k-1} is a polynomial of degree $k-2$ and $R_k(s,n)$ is the remainder term of the Taylor series. We have a Taylor estimate $R_k(s,n) \leq C_k/n^k$ for $|s| \leq k+1$ and therefore:

$$\zeta_g(s) - \zeta_f(s) = s\sum_{n\geq 1}\frac{f(n)}{n^s}\bigg(-\frac{1}{n}+\cdots+\frac{P_{k-1}}{n^{k-1}}+R_k(s,n)-\frac{1}{n}+\cdots+\frac{P_{k-1}}{n^{k-1}}+R_k(s,n)\bigg).$$

By our estimates, this series is absolutely convergent in a neighbourhood of $s = k$ and therefore s divides $\zeta_g(s) - \zeta_f(s)$ as analytic functions. Therefore, $\zeta_g(0) = \zeta_f(0)$ as required.

□

6. SOME APPLICATIONS TO DIRICHLET L-FUNCTIONS

Our work so far already gives us some interesting applications to Dirichlet L-functions. The most basic one is to rationality of the values of L-functions.

6.1. **Rationality of the values of L-functions.**

Let $L(s, \chi)$ be a Dirichlet L-function. Note that $\chi(n)$ is periodic and hence satisfies a linear recurrence. Therefore, $L(-k, \chi)$ for integral $k \geq 0$ can be evaluated using our divergent summation techniques.

In particular, let $\mathbb{Q}(\chi)$ be the field in which χ takes values. If f is a function taking values in k, we know that $\Sigma(f)$ is also in k. Therefore we immediately have:

Theorem 9. *For integral $k \geq 0$ and Dirichlet Character χ, we have $L(-k, \chi) \in \mathbb{Q}(\chi)$.*

In particular, $\zeta(-k) \in \mathbb{Q}$.

6.2. **Compatibility with Galois action.**

Since $L(-k, \chi)$ are algebraic integers, a natural question is to study the action of the Galois group $\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$ on these values. In fact, this is particular easy. Define $\chi^\sigma(n) = \sigma(\chi(n))$. Then, we have the theorem:

Theorem 10. *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$ be a Galois automorphism. Then, for integral $k \geq 0$:*

$$L(-k, \chi^\sigma) = L(-k, \chi)^\sigma.$$

Proof. Let us take our base field to be $k = \mathbb{Q}(\chi)$. For $f \in X$, define $\Sigma^\sigma(f) = \sigma^{-1}(\Sigma(\sigma(f)))$. Note that the theorem is equivalent to showing that $\Sigma^\sigma = \Sigma$. To do this, we only need to show that it satisfies the properties characterizing Σ. But this is easy:

For compact sequences f, say bounded by N, we have:

$$\Sigma^\sigma(f) = \sigma^{-1}\bigg(\sum_{n\leq N}\sigma(f(n))\bigg) = \sum_{n\leq N} f(n) = \Sigma(f).$$

For invariance under translation and dilation, note that $S[f^\sigma] = S[f]^\sigma$ and similarly for D_m. Also, $X^{\neq 1}$ is invariant under Σ after we extend it to an automorphism of $\overline{\mathbb{Q}}$. Therefore, for $f \in X^{\neq 1}$, we have:

$$\Sigma^\sigma(S[f]) = \sigma^{-1}(\Sigma(S[f^\sigma])) = \sigma^{-1}(\Sigma(f)^\sigma) = \Sigma^\sigma(f).$$

An exactly similar proof holds for D_m.

□

7. Summation in non-archimedean fields

Suppose now that k is a field equipped with a non archimedean valuation $|\cdot|$. The examples we will be interested in will be $\mathbb{Z}_p$ and $k((t))$ with the usual valuations. Let R be the ring of integers of k.

In this case, we can say quite a bit more about the Σ operator. A natural question to ask in this case is whether $\Sigma : X \longrightarrow k$ is a continuous operator where X is given the uniform topology. Unfortunately, this is too much to ask for as the following example shows:

Example 5. *Let $f_k(n) = \alpha_k^n$ where $\alpha_k \neq 1$ for all $k \in \mathbb{N}$ but $\lim \alpha_k = 1$. Then, one can verify that $\Sigma[f_k] = \alpha_k/(1-\alpha_k)$ while $\Sigma[(1,1,\dots)] = -1/2$ and therefore $\lim \Sigma[f_k] \neq \Sigma[\lim(f_k)]$.*

Nevertheless, Σ is continuous for an important subspace of X as we now show:

Theorem 11. *Let $\Lambda \subset X$ be defined by:*

$$\Lambda = \{f : \mathbb{N} \longrightarrow R \in X^\alpha : |\alpha - 1| = 1\}.$$

That is, Λ consists of those sequences taking values in R for which the α are such that $\alpha - 1$ is a unit. Then, for $f, g \in \Lambda$, one has

$$|\Sigma[f] - \Sigma[g]| \leq \Sigma[f-g].$$

Proof. The proof is quite easy. Note that $\Lambda \subset X^{\neq 1}$. Recall from the remarks following Theorem 4 that $\Sigma[f] = \Sigma[f^c]/P_f(1)$. Here, P_f is the characteristic polynomial of S on X/Y^c. As such, it's roots are the exponents α of f and by definition of Λ, $|P_f(1)| = 1$.

Moreover, $f_c = P_f(S)[f]$ and since the roots of P_f are in R, we have $P_f \in R[X]$ and so $f_c(n) \in f(\mathbb{N}) = R$. Together, this implies that $\Sigma[f] \in R$.

Let a uniformizer of R be denoted π. Now, suppose that $f = g + \pi^n h$ for $h : \mathbb{N} \longrightarrow R$ and consequently in Λ. Then, $\Sigma[f] = \Sigma[g] + \pi^n \Sigma[h]$ by linearity. However, by the previous paragraph, we know that $\Sigma[h] \in R$ and therefore $\pi^n | (\Sigma[f] - \Sigma[g])$ which proves our result.

□

This theorem will be at the root of most of what follows and it's importance cannot be understated!

8. Generalized Kummer Congruences

We have finally set up enough general theory to quickly prove the Kummer congruences using the method outlined in the introduction. Let us first assume that the L-functions under consideration have non trivial Dirichlet characters.

Let us first prove a preliminary version of the Kummer congruences for Dirichlet L-functions with non trivial characters that exposes most of the ideas in the proof.

Theorem 12. *Let $\chi_1, \chi_2 : \mathbb{Z} \longrightarrow K$ be two non trivial Dirichlet characters (with possibly distinct conductors N_1, N_2). Also, let $\mathfrak{p}$ be a prime of K lying over a rational prime p not dividing the conductors of χ_1, χ_2 such that for all n coprime to p and two non negative integers k, l, we have:*

$$\chi_1(n) n^k \equiv \chi_2(n) n^l \pmod{\mathfrak{p}^r}.$$

Then, the corresponding L-functions have integral values:

$$L(-k, \chi_1), L(-l, \chi_2) \in \mathscr{O}_{K_\mathfrak{p}}$$

and satisfy the congruence:

$$L(-k, \chi_1) \equiv L(-l, \chi_2) \pmod{\mathfrak{p}^r}.$$

Proof. Define $f_1(n) = \chi_1 n^k, f_2(n) = \chi_2(n)$. The strategy of the proof is simple. We would like to modify f_1, f_2 simultaneously so that we can apply Theorem 11 from the previous section.

A problem with applying the theorem is the following: While $f_1(n) \equiv f_2(n) \pmod{\mathfrak{p}^r}$ for n coprime to p, this might not be true for n divisible by p.

To fix this, define

$$\theta_p(n) = \begin{cases} 1 & p \nmid n \\ 0 & p | n \end{cases}$$

and:

$$g_1(n) = \theta_p(n) f_1(n), g_2(n) = \theta_p(n) f_2(n).$$

It is now clear that $g_1(n) \equiv g_2(n) \pmod{\mathfrak{p}^r}$ for all n.

To apply Theorem 11, we also need to check that the exponents α of $g_1 - g_2$ are such that $\alpha - 1$ is a unit in $K_{\mathfrak{p}}$. The exponents of $\theta_p(n)$ are p-roots of unity while the exponents of f_1, f_2 are non trivial N_1, N_2- roots of unity (see example 4, section 4).

Since N_1, N_2 are coprime to p and the exponents of $f(n)g(n)$ are the products of exponents of f and g, it is now clear that the exponents of $g_1 - g_2$ satisfy the required property.

Applying Theorem 11 to $g_1 - g_2$, we have:

$$\Sigma(g_1) \equiv \Sigma(g_2) \pmod{\mathfrak{p}^r}.$$

To calculate $\Sigma(g_i)$, note that:

$$\theta_p(n)\chi(n)n^k = \chi(n)n^k - \chi(p)p^k D_p[\chi(n)n^k].$$

Applying Σ to both sides and using invariance under D_p, we have:

$$\Sigma(g_1) = (1 - \chi_1(p)p^k)\Sigma(\chi(n)n^k)$$

and similarly for g_2. Therefore, we have:

$$(1 - \chi_1(p)p^k)L(-k, \chi_1) \equiv (1 - \chi_2(p)p^l)L(-l, \chi_2) \pmod{\mathfrak{p}^r}.$$

Finally, since $\chi_1(p)p^k \equiv \chi_2(p)p^l \pmod{\mathfrak{p}^r}$, we can cancel these factors to obtain:

$$L(-k, \chi_1) \equiv L(-l, \chi_2) \pmod{\mathfrak{p}^r}.$$

Also, applying Theorem 11 to g_1, g_2 individually gives us the integrality condition:

$$L(-k, \chi_1), L(-l, \chi_2) \in \mathscr{O}_{K_{\mathfrak{p}}}.$$

□

A small modification of the above proof will also prove a similar theorem for the Riemann zeta function. We will use the same notation as in the proof of the previous theorem.

Theorem 13. *Let $\chi_1, \chi_2 : \mathbb{Z} \longrightarrow K$ be two Dirichlet characters (possibly both trivial) of conductor N_1, N_2. As before, let $\mathfrak{p}$ be a prime of K lying over the rational prime p (but now with no restrictions on p). Let k, l be non negative integers such that for all n coprime to p, we have:*

$$\chi_1(n)n^k \equiv \chi_2(n)n^l \pmod{\mathfrak{p}^r}.$$

Then, for any q coprime to pN_1N_2, the L-functions satisfy the integrality condition:

$$(1 - \chi_1(q)q^{k+1})(1 - \chi_1(p)p^k)L(-k, \chi_1), (1 - \chi_2(q)q^{l+1})(1 - \chi_2(p)p^l)L(-l, \chi_2) \in \mathscr{O}_{K_{\mathfrak{p}}}$$

and the congruence:

$$(1 - \chi_1(q)q^{k+1})(1 - \chi_1(p)p^k)L(-k, \chi_1) \equiv (1 - \chi_2(q)q^{l+1})(1 - \chi_2(p)p^l)L(-l, \chi_2) \pmod{\mathfrak{p}^r}.$$

This theorem will specialize to both the usual Kummer congruences as well the Von Staudt-Clausen theorem as we will show later.

Proof. As before, let $f_1(n) = \chi_1(n)n^k$ and $f_2(n) = \chi_2(n)n^l$. The strategy of the proof is the same as before. The additional wrinkle in this case is that if $\chi(n) = 1$ or the conductor is divisible by p, then the exponents α of $\theta_p(n)\chi(n)n^k$ will not be such that $\alpha - 1$ is a unit in $K_\mathfrak{p}$.

To fix this, for q coprime to p, define:

$$\begin{aligned} h_1 = f_1 - \chi_1(q)q^{k+1}D_q[f_1] &= \chi_1(n)n^k - (1 + \zeta_q^n + \zeta_q^{2n} + \cdots + \zeta_q^{(q-1)n})\chi_1(n)n^k \\ &= -(\zeta_q^n + \zeta_q^{2n} + \cdots + \zeta_q^{(q-1)n})\chi_1(n)n^k \end{aligned}$$

and similarly for h_2.The above expression also shows that the exponents of h_i are non trivial qN_i- roots of unity. Note also that under our hypothesis:

$$h_1(n) \equiv h_2(n) \pmod{\mathfrak{p}^r}.$$

Note also that:

$$\Sigma(h_1) = (1 - \chi_1(q)q^{k+1})\Sigma(f_1)$$

and similarly for h_2. This done, the rest of the proof proceeds much the same as before.

Therefore, define:

$$g_i(n) = \theta_p(n)f_i(n)$$

and once again, we have:

$$g_1(n) \equiv g_2(n) \pmod{\mathfrak{p}^r}$$

and the exponents α of $g_1(n) - g_2(n)$ are such that $\alpha - 1$ is a unit. We can also calculate $\Sigma(g_i)$ as before:

$$\theta_p h_1 = h_1 - \chi_1(p)p^k D_p[h_1]$$

since $\zeta_q^{rn} \longrightarrow \zeta_q^{rn/p}$ simply permutes the non trivial q-roots of unity. Applying Σ:

$$\Sigma(g_1) = (1 - \chi_1(p)p^k)\Sigma(h_1) = (1 - \chi_1(p)p^k)(1 - \chi_1(q)q^{k+1})\Sigma(f_1).$$

Applying Theorem 11 to g_1, g_2 individually gives us the integrality condition:

$$(1 - \chi_1(q)q^{k+1})(1 - \chi_1(p)p^k)L(-k, \chi_1), (1 - \chi_2(q)q^{l+1})(1 - \chi_2(p)p^l)L(-l, \chi_2) \in \mathscr{O}_{K_\mathfrak{p}}$$

and applying it to $g_1 - g_2$ gives us:

$$(1 - \chi_1(q)q^{k+1})(1 - \chi_1(p)p^k)L(-k, \chi_1) \equiv (1 - \chi_2(q)q^{l+1})(1 - \chi_2(p)p^l)L(-l, \chi_2) \pmod{\mathfrak{p}^r}.$$

□

We can specialize to get useful theorems:

Corollary 14 (Generalized Kummer Congruences)**.** *Suppose $\chi_1(q)q^{k+1}$ and $\chi_2(q)q^l$ are coprime to p. Then:*

$$(1 - \chi_1(p)p^k)L(-k, \chi_1) \equiv (1 - \chi_2(p)p^l)L(-l, \chi_2) \pmod{\mathfrak{p}^r}.$$

In particular, if $\chi_1 = \chi_2 \equiv 1$ and $(p-1) \nmid (k+1)(l+1)$ and $k \equiv l \pmod{(p-1)p^{r-1}}$, then:

$$(1 - p^k)\zeta(-k) \equiv (1 - p^l)\zeta(-l) \pmod{p^r}.$$

This final expression is the standard form for the Kummer congruences.

Corollary 15 (Kummer Congruences with Teichmuller characters)**.** *Let* $\omega : (\mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathbb{Z}(\mu_{p-1})$ *be the Teichmuller character. That is,* $\omega(n)$ *is the unique p-th root of unity such that* $\omega(n) \equiv n \pmod{p}$*. Let* $\mathfrak{p}$ *be a prime of* $\mathbb{Q}(\mu_{p-1})$ *lying over* p*. Then for non negative integers* k, l *such that* $k+1$ *is not divisble by* $p-1$*, we have:*

$$\zeta(-k-l) \equiv L(-k, \omega^l) \pmod{\mathfrak{p}}.$$

Specializing even more:

$$\zeta(1-k) \equiv L(-1, \omega^{k-2}) \pmod{\mathfrak{p}}.$$

Remark 16. This final form is used in Ribet's paper on the converse to Herbrand-Ribet to prove congruences between modular forms of different level (and weight) and plays a cruical role in the proof. Ribet proves the congruence using an explicit representation of $L(-1, \epsilon)$ in terms of Bernoulli numbers.

Suppose again that $\chi_1 = \chi_2 \equiv 1$. We will show in the next section that $\zeta(-k) = B_{k+1/(k+1)}$ where B_k is the k-th Bernoulli number. We can therefore use the above results to gain p-adic information about the Bernouli numbers.

Corollary 17 (p-adic Valuation of the Bernoulli numbers)**.** *The denominator of* B_k *can only be divisible by a prime* p *to the first power. Further, this can only happen if* $(p-1)|n$*.*

Proof. Let p^{r-1} be the highest power of p that divides $k+1$. That is, $v_p(k+1) = r-1$. By the above, we know that for any q coprime to p:

$$(q^{k+1}-1)\zeta(-k) = (q^{k+1}-1)\frac{B_{k+1}}{k+1} \in \mathbb{Z}_p.$$

First suppose that $(p-1) \nmid (k+1)$. Choose q so that $q^{k+1} \not\equiv 1 \pmod{p}$. Taking valuations, this shows:

$$v_p(B_{k+1}) \geq v_p(k+1) = r \geq 0.$$

Next, suppose that $(p-1)|(k+1)$. In this case, choose q so that $v_p(q^{k+1}-1) = r$. This can be done by taking q to be a cyclic generator of the cyclic group $(\mathbb{Z}/p^{r+1}\mathbb{Z})^{\times}$. Taking valuations:

$$\begin{aligned} v_p(q^{k+1}-1) + v_p(B_{k+1}) - v_p(k+1) &\geq 0 \\ \implies r + v_p(B_{k+1}) - (r-1) &\geq 0 \\ \implies v_p(B_{k+1}) &\geq -1 \end{aligned}$$

□

Remark 18. In fact, in the case that $p-1|(k+1)$, the denominator is actually divisible by p and moreover, the numerator is coprime to p. In other words, $v_p(\zeta(-k)) = 1 + v_p(k+1)$ in this case. It might be interesting to prove this using the techniques here.

9. Explicit computation of special values of Dirichlet L-functions

This is the only entirely original section of this note. I will compute the explicit values of Dirichlet L-functions in terms of the Bernoulli numbers. Let us first begin with Dirichlet L-functions with non trivial character.

Theorem 19. *For $\chi : \mathbb{Z}/f\mathbb{Z} \longrightarrow \mathbb{C}^\times$ a non trivial Dirichlet character and k a non negative integer, we have:*

$$L(-k,\chi) = -\frac{B_{k+1,\chi}}{k+1}$$

where the $B_{k,\chi}$ are defined by the exponential generating function:

$$\sum_{a=1}^{f} \chi(a)\frac{te^{at}}{e^{ft}-1} = \sum_{k=0}^{\infty} B_{k,\chi}\frac{t^k}{k!}.$$

Proof. The idea is simple, as outlined in the introduction. Let us take our base field $k = \mathbb{Q}(\chi)((t))$. This is a non archimedean complete field in a natural way with ring of integers $R = \mathbb{Q}(\chi)[[t]]$ and the results of section 7 are applicable.

In particular, recall that Λ is the set of sequences taking values in R and for which the exponents α are such that $\alpha - 1$ is a unit. Then, we showed that $\Sigma : \Lambda \longrightarrow R$ is a continuous functional.

In particular, for a sequence $g_k \in \Lambda$ such that $\lim_k g_k = g \in \Lambda$, we have $\lim_k \Sigma(g_k) = \Sigma(g)$. Let us take

$$g_k(n) = \sum_{i=0}^{k} n^i \chi(n)\frac{t^i}{i!}.$$

It is easy to see that $g = \lim_k g_k = \chi(n)e^{nt} \in \Lambda$. This is because the exponents associated to $\chi(n)$ are all roots of unity and not equal to 1 (see example 4, section 4) and the exponents of g are therefore of the form $e^t\zeta_f$ for ζ_f a non trivial f-th root of unity of 1. Certainly, $e^t\zeta_f - 1$ is a unit in R.

However, it is also easy to compute the action of Σ on g_k and g. By definition, we have:

$$\Sigma(g_k) = \sum_{i=0}^{k} L(-i,\chi)\frac{t^i}{i!}$$

and therefore:

$$\lim_k \Sigma(g_k) = \sum_{i=0}^{\infty} L(-i,\chi)\frac{t^i}{i!}.$$

On the other hand:

$$\Sigma(g) = \sum_{a=1}^{f} \chi(a)\frac{e^{at}}{1-e^{ft}} = \frac{1}{t}\sum_{k=0}^{\infty} -B_{k,\chi}\frac{t^k}{k!}.$$

Since $\Sigma(g) = \lim \Sigma(g_k)$, on equation the coefficients of their respective power series expansions, we have:

$$L(1-k,\chi) = -\frac{B_{k,\chi}}{k}.$$

□

If we try to adapt the above proof to the case of the Riemann zeta function by letting χ be the trivial function, we come across the problem that $g(n) = \lim_k g_k(n) = e^{nt}$ is not in Λ. The corresponding α is just e^t and certainly, $e^t - 1$ is not a unit in R.

We came across a similar problem while proving the Kummer congruences and we will solve this in a similar way.

Instead of calculating $\zeta(-k)$, let us instead calculate $(1-2^{k+1})\zeta(-k)$. The corresponding series for this is:

$$f_i(n) = n^i - 2^{i+1}D_2[n^i] = -(-1)^n n^i.$$

Clearly, $f_k(n)$ has exponent -1 and $-1-1=2$ is a unit in R.

As before, let us define:

$$g_k(n) = \sum_{i=0}^{k} -(-1)^i n^i \frac{t^i}{i!}.$$

Now, $g = \lim_k g_k = -(-1)^n e^{nt}$. This is in Λ since the exponent is $-e^t$. Repeating the calculations as before will show:

$$\sum_{k\geq 0}(1-2^{k+1})\zeta(-k)\frac{t^k}{k!} = \frac{e^t}{1+e^t}.$$

It is now a simple verification that $\zeta(-k) = -\frac{B_{k+1}}{k+1}$ satisfies the above equation. Indeed, substituting the above value for $\zeta(-k)$ and using the exponential generating function for B_k, we have:

$$\begin{aligned}\sum_{k\geq 0}(1-2^{k+1})\zeta(-k)\frac{t^k}{k!} &= \sum_{k\geq 0} -(1-2^{k+1})B_{k+1}\frac{t^k}{(k+1)!} \\ &= -\frac{1}{t}\left(\frac{t}{1-e^{-t}}-1\right) + \frac{1}{t}\left(\frac{2t}{1-e^{-2t}}-1\right) \\ &= \frac{1}{1+e^{-t}}\end{aligned}$$

and we have therefore proven:

Theorem 20. *For k a non negative integer and B_k the Bernoulli Number, we have:*

$$\zeta(-k) = -\frac{B_{k+1}}{k+1}.$$